<u>Some proofs on CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality)</u>

Yue Kwok Choy

1. **(a)** Show that the quadratic function $f(x) = Ax^2 + 2Bx + C$, where A, B, C $(A \neq 0)$ are real, is non-negative for all real value of x if and only if $A > 0$ and $B^2 \leq AC$.

(b) Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be two sets of real numbers, prove that

$$\left[\sum_{i=1}^{n} a_i b_i\right]^2 \leq \left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right] .$$

(c) Discuss the necessary and sufficient conditions for equality holds in **(b)** .

2. **(a)** Prove that $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} b_i^2 = 1$, then $\left[\sum_{i=1}^{n} a_i b_i\right]^2 \leq 1$, where $a_i , b_i \in \mathbb{R}$.

(b) Prove that $\left[\sum_{i=1}^{n} a_i b_i\right]^2 \leq \left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right]$, where $a_i , b_i \in \mathbb{R}$

and the equality holds iff $\frac{a_i}{b_i}$ is the same for $i = 1$ to n .

3. **(a)** Show that $\left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right] - \left[\sum_{i=1}^{n} a_i b_i\right]^2 = \sum_{1 \leq i \neq j \leq n} \left(a_i b_j - a_j b_i\right)^2$.

(b) Show that $\left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} b_i^2\right] \geq \left[\sum_{i=1}^{n} a_i b_i\right]^2$, where $a_i , b_i \in \mathbb{R}$

and the equality holds iff $\frac{a_i}{b_i}$ is the same for $i = 1$ to n .

4. **(a)** Prove that $\frac{(a+b)^2}{x+y} \leq \frac{a^2}{x} + \frac{b^2}{y}$, where $a, b \in \mathbb{R}$, $x, y \in +\mathbb{R}$ 。

(b) Prove that $\frac{(a_1 + a_2 + \ldots + a_n)^2}{x_1 + x_2 + \ldots + x_n} \leq \frac{a_1^2}{x_1} + \frac{a_2^2}{x_2} + \ldots + \frac{a_n^2}{x_n}$, where $a_k \in \mathbb{R}$, $x_k \in +\mathbb{R}$ 。

(c) By choosing suitable a_k , x_k in **(b)** , prove that $\left[\sum_{k=1}^{n} \alpha_k^2\right]\left[\sum_{k=1}^{n} \beta_k^2\right] \geq \left[\sum_{k=1}^{n} \alpha_k \beta_k\right]^2$,

where $\alpha_k , \beta_k \in \mathbb{R}$ 。

(d) Discuss the necessary and sufficient conditions for equality holds in **(c)** .

5. Given that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two sets of non-zero real numbers,

(a) Shat that $\forall \lambda \neq 0$,

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \frac{1}{2}\left[\lambda^2 \sum_{k=1}^{n} a_k{}^2 + \frac{1}{\lambda^2}\sum_{k=1}^{n} b_k{}^2\right] .$$

(b) Hence, by choosing suitable λ, deduce that

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2 \le \left[\sum_{k=1}^{n} a_k{}^2\right]\left[\sum_{k=1}^{n} b_k{}^2\right].$$

Find also a necessary and sufficient condition for the equality sign holds.

Solution

5. **(a)** $\left(\lambda a_k \pm \frac{1}{\lambda} b_k\right)^2 \ge 0, \quad \forall \lambda \neq 0, \quad k = 1,2,\ldots,n.$

$$\lambda^2 a_k{}^2 \pm 2a_k b_k + \frac{1}{\lambda^2} b_k{}^2 \ge 0$$

$$\lambda^2 \sum_{k=1}^{n} a_k{}^2 \pm 2\sum_{k=1}^{n} a_k b_k + \frac{1}{\lambda^2}\sum_{k=1}^{n} b_k{}^2 \ge 0$$

$$\pm \sum_{k=1}^{n} a_k b_k = \frac{1}{2}\left[\lambda^2 \sum_{k=1}^{n} a_k{}^2 + \frac{1}{\lambda^2}\sum_{k=1}^{n} b_k{}^2\right]$$

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \frac{1}{2}\left[\lambda^2 \sum_{k=1}^{n} a_k{}^2 + \frac{1}{\lambda^2}\sum_{k=1}^{n} b_k{}^2\right]$$

(b) Let $\lambda^2 = \sqrt{\sum_{k=1}^{n} b_k{}^2 \Big/ \sum_{k=1}^{n} a_k{}^2}$ Since $\lambda \neq 0$, by **(a)** we have ,

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \frac{1}{2}\left[\left(\sqrt{\sum_{k=1}^{n} b_k{}^2 \Big/ \sum_{k=1}^{n} a_k{}^2}\right)\sum_{k=1}^{n} a_k{}^2 + \frac{1}{\sqrt{\sum_{k=1}^{n} b_k{}^2 \Big/ \sum_{k=1}^{n} a_k{}^2}}\sum_{k=1}^{n} b_k{}^2\right]$$

$$= \sqrt{\left[\sum_{k=1}^{n} a_k{}^2\right]\left[\sum_{k=1}^{n} b_k{}^2\right]}$$

$$\therefore \quad \left[\sum_{k=1}^{n} a_k b_k\right]^2 \le \left[\sum_{k=1}^{n} a_k{}^2\right]\left[\sum_{k=1}^{n} b_k{}^2\right]$$

Equality holds $\Leftrightarrow \quad \lambda a_k \pm \frac{1}{\lambda} b_k = 0, \forall k = 1,2,\ldots,n \quad \Leftrightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n}$